Differential Equations Eulers [30 marks]

1. [Maximum mark: 3] 26M.2.AHL.TZ3.12

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - \frac{y}{x}$, for $x, y > 0$. It is given that $y = 2$ when $x = 1$.

(a) Use Euler's method with a step length of 0.01 to find an approximate value of y when $x = 1.05$. [3]

Markscheme

attempt to use the recursive formula

$y_n = y_{n-1} + 0.01\left({y_{n-1}}^3 - \frac{y_{n-1}}{x_{n-1}}\right)$ ***(M1)***

n	x_n	y_n
0	1	2
1	1.01	2.06
2	1.02	2.12702…
3	1.03	2.20240…
4	1.04	2.28784…
5	1.05	2.38559…

$y_1 = 2.06$ ***(A1)***

$y_5 = 2.39$ ***A1***

[3 marks]

2. [Maximum mark: 4] 25N.2.AHL.TZ3.12

Consider the following homogenous differential equation

$\left(x^2+xy\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x^2+xy-3y^2$ where $x>0$ and $y>\frac{x}{2}$.

It is given that $y=\frac{3}{2}$ when $x=1$.

(a.i) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=1$. [1]

Markscheme

$\frac{\mathrm{d}y}{\mathrm{d}x}=-1.7$ ***A1***

[2 marks]

(a.ii) Use Euler's method with two equal steps to estimate the value of y when $x=1.4$. [3]

Markscheme

attempt to use Euler's formula with step length 0.2 ***(M1)***

step	x	y
0	1	1.5
1	1.2	1.16
2	1.4	1.07491524

$1.5+0.2\times-1.7$ **OR** 1.16 ***(A1)***

1.07 ***A1***

Note: Accept 1.075.

[2 marks]

3. [Maximum mark: 3] 25M.2.AHL.TZ1.9

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2+y}$.

The solution curve passes through the point $(1,\ 0)$.

(a) Use Euler's method with a step value of 0.25 to estimate the value of y when $x = 2$. [3]

Markscheme

attempt to use Euler's method with a step of 0.25 ***(M1)***

$(x_{n+1} = x_n + 0.25),\ y_{n+1} = y_n + 0.25\left(\frac{2x_n}{{x_n}^2+y_n}\right)$

$(y_0 = 0)$

$y_1 = 0.5$

$y_2 = 0.803030\ldots$ ***(A1)***

$y_3 = 1.04868\ldots$

Note: Award ***(A1)*** for at least two correct intermediate values given to 3sf.

If $x = 2,\ y = 1.26152\ldots$

$y = 1.26$ ***A1***

[3 marks]

4. [Maximum mark: 3] 24M.2.AHL.TZ2.12

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - y \operatorname{cosec} 2x = \sqrt{\tan x}$, where $0 < x < \frac{\pi}{2}$ and $y = \frac{\pi}{4}$ at $x = \frac{\pi}{4}$.

(a) Use Euler's method with step length $\frac{\pi}{12}$ to find an approximate value of y when $x = \frac{5\pi}{12}$.

Give your answer correct to three significant figures. [3]

Markscheme

attempts to use Euler's method ***(M1)***

$x_{n+1} = x_n + \frac{\pi}{12}$; $y_{n+1} = y_n + \frac{\pi}{12} \times \frac{\mathrm{d}y}{\mathrm{d}x}$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = y \operatorname{cosec} 2x + \sqrt{\tan x}$

$y_1 = 1.25281\ldots \left(= \frac{\pi}{4} + \frac{\pi}{12}\left(\frac{\pi}{4} + 1\right)\right)$ ***(A1)***

$y_2 = 1.97608\ldots$

$y = 1.98$ ***A1***

[3 marks]

5. [Maximum mark: 3] 23N.2.AHL.TZ1.9

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4-y}{10}$, where $y = 2$ when $x = 0$.

(a) Use Euler's method with a step size of 0.1 to find an approximation for y when $x = 0.5$. Give your answer correct to four significant figures. [3]

Markscheme

attempt to use recursive formula $y_n = y_{n-1} + 0.1\left(\frac{4-y_{n-1}}{10}\right)$

n	x_n	y_n
0	0	2
1	0.1	2.02
2	0.2	2.0398
3	0.3	2.05940…
4	0.4	2.07880…
5	0.5	2.09801…

$y_1 = 2.02$ ***(A1)***

$y_5 = 2.098$ ***A1***

Note: Accept any answer which rounds to the correct 4sf value.

Award no marks for a final answer of 2.1 or 2.10 with no working.

[3 marks]

6. [Maximum mark: 14] 22M.2.AHL.TZ1.12

Consider the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2x^2$ for $x > 0$ and $y > 2x$. It is given that $y = 3$ when $x = 1$.

(a) Use Euler's method, with a step length of 0.1, to find an approximate value of y when $x = 1.5$. [4]

Markscheme

attempt to use Euler's method ***(M1)***

$x_{n+1} = x_n + 0.1$; $y_{n+1} = y_n + 0.1 \times \frac{\mathrm{d}y}{\mathrm{d}x}$, where $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2x^2}{x^2}$

correct intermediate y-values ***(A1)(A1)***

$3.7, 4.63140\ldots, 5.92098, 7.79542\ldots$

Note: ***A1*** for any two correct y-values seen

$y = 10.6958\ldots$

$y = 10.7$ ***A1***

Note: For the final ***A1***, the value 10.7 must be the last value in a table or a list, or be given as a final answer, not just embedded in a table which has further lines.

[4 marks]

(b) By solving the differential equation, show that $y = \frac{8x + x^4}{4 - x^3}$. [10]

Markscheme

attempt to separate variables v and x ***(M1)***

$\int \frac{\mathrm{d}v}{v^2 - v - 2} = \int \frac{\mathrm{d}x}{x}$

$\int \frac{\mathrm{d}v}{(v-2)(v+1)} = \int \frac{\mathrm{d}x}{x}$ ***(A1)***

attempt to express in partial fraction form ***M1***

$\frac{1}{(v-2)(v+1)} \equiv \frac{A}{v-2} + \frac{B}{v+1}$

$\frac{1}{(v-2)(v+1)} = \frac{1}{3}\left(\frac{1}{v-2} - \frac{1}{v+1}\right)$ **A1**

$\frac{1}{3}\int\left(\frac{1}{v-2} - \frac{1}{v+1}\right)\,\mathrm{d}\,v = \int \frac{\mathrm{d}x}{x}$

$\frac{1}{3}(\ln|v-2| - \ln|v+1|) = \ln|x|(+c)$ **A1**

Note: Condone absence of modulus signs throughout.

EITHER

attempt to find c using $x = 1,\ y = 3,\ v = 3$ **M1**

$c = \frac{1}{3}\ln\frac{1}{4}$

$\frac{1}{3}(\ln|v-2| - \ln|v+1|) = \ln|x| + \frac{1}{3}\ln\frac{1}{4}$

expressing both sides as a single logarithm **(M1)**

$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(\frac{|x|^3}{4}\right)$

OR

expressing both sides as a single logarithm **(M1)**

$\ln\left|\frac{v-2}{v+1}\right| = \ln\left(A|x|^3\right)$

attempt to find A using $x = 1,\ y = 3,\ v = 3$ **M1**

$A = \frac{1}{4}$

THEN

$\left|\frac{v-2}{v+1}\right| = \frac{1}{4}x^3$ (since $x > 0$)

substitute $v = \frac{y}{x}$ (seen anywhere) **M1**

$\frac{\frac{y}{x}-2}{\frac{y}{x}+1} = \frac{1}{4}x^3$ (since $y > 2x$)

$\left(\Rightarrow \frac{y-2x}{y+x} = \frac{1}{4}x^3\right)$

attempt to make y the subject **M1**

$y - \frac{x^3y}{4} = 2x + \frac{x^4}{4}$ **A1**

$y = \frac{8x+x^4}{4-x^3}$ **AG**

[10 marks]

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